# Integer programming

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**Definition 1.** Algorithms which change the boundary of the solution region in order to find the optimal solution of an integer programme are called *cut algorithms*. The branch-and-bound algorithm does this by splitting the solution region into two and then discard the one which does not contain the optimal solution. The Gomory algorithm, on the other hand, reduce the feasible region with the help of a new constraint without the region being splitted.

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**Definition 2.** We call *branching* a process by which a programme whose solution contains a non-integral $j < x_i < k$ is made into two separate programmes having the additional constraint $x_i \leq j$ in one, and $x_i \geq k$ in the other, the objective together with all the constraints of the original problem of which remain the same. Here $j$ and $k$ are positive integers and $j < k$.

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**Definition 3.** In the branch-and-bound algorithm, if the objective is maximisation, the value of the objective obtained when the first integral approximation occurs is said to be the lower bound for the problem, and if the objective is minimisation it is said to be the upper bound of the same.

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**Algorithm 1** *Branch-and-bound algorithm for integer programming.*

**find** first approximation
**while** approximations not all integers **do**
  **choose** $x_i$ from all non-integral variables such that $\min\left(|x_i - \lfloor x_i \rfloor|, |x_i - \lceil x_i \rceil|\right)$
    is maximised
  **branch**
  **choose** the branch whose value of the objective is maximum
**endwhile**
solution$\leftarrow$last approximation

**Example 1.** (*Problem 6.9; Bronson, 1982*)

$$\begin{aligned} \text{maximise:} \quad & z = x_1 + 2x_2 + x_3 \\ \text{subject to:} \quad & 2x_1 + 3x_2 + 3x_3 \leq 11 \\ \text{with:} \quad & \text{all variables non-negative and integral} \end{aligned}$$

**Solve** by branch-and-bound algorithm.

**Solution.**Draw a simplex table of Programme 1.

| | | $x_1$<br>1 | $x_2$<br>2 | $x_3$<br>1 | $x_4$<br>0 | |
|---|---|---|---|---|---|---|
| $x_4$ | 0 | 2 | 3 | 3 | 1 | 11 |
| | | -1 | -2 | -1 | 0 | 0 |

Replace $x_4$ for $x_2$ as the basic variable.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | |
|---|---|---|---|---|---|
| $x_2$ | $\frac{2}{3}$ | 1 | 1 | $\frac{1}{3}$ | $1\frac{1}{3}$ |
| | $\frac{1}{3}$ | 0 | 1 | $\frac{2}{3}$ | $\frac{22}{3}$ |

$x_2^* = 1\frac{1}{3} = 3.6$, $x_1^* = x_3^* = x_4^* = 0$, $z^* = \frac{22}{3}$ Since $3 < x_2^* < 4$, branch into two programmes, namely Programme 1 where $x_2 \leq 3$, and Programme 2 where $x_2 \geq 4$. Consider first Programme 2.

$$\begin{aligned} \text{maximise:} \quad & z = x_1 + 2x_2 + x_3 \\ \text{subject to:} \quad & 2x_1 + 3x_2 + 3x_3 \leq 11 \\ & x_2 \leq 3 \\ \text{with:} \quad & \text{all variables non-negative and integral} \end{aligned}$$

Use the simplex method in a tabulated form.

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 1 | 0 | 0 | |
| $x_4$ | 0 | 2 | 3 | 3 | 1 | 0 | 11 |
| $x_5$ | 0 | 0 | 1 | 0 | 0 | 1 | 3 |
| | | -1 | -2 | -1 | 0 | 0 | 0 |

Replace the basic variable $x_5$ with $x_2$.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | |
|---|---|---|---|---|---|---|
| $x_4$ | 2 | 0 | 3 | 1 | -3 | 2 |
| $x_2$ | 0 | 1 | 0 | 0 | 1 | 3 |
| | -1 | 0 | -1 | 0 | 2 | 6 |

Replace the basic variable $x_4$ with $x_1$.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | |
|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | $\frac{3}{2}$ | $\frac{1}{2}$ | $-\frac{3}{2}$ | 1 |
| $x_2$ | 0 | 1 | 0 | 0 | 1 | 3 |
| | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 7 |

$x_1^* = 1$, $x_2^* = 3$, $x_3^* = x_4^* = x_5^* = 0$, $z^* = 7$ Then consider Programme 3.

$$\begin{aligned}
\text{maximise:} \quad & z = x_1 + 2x_2 + x_3 \\
\text{subject to:} \quad & 2x_1 + 3x_2 + 3x_3 \leq 11 \\
& x_2 \geq 4 \\
\text{with:} \quad & \text{all variables non-negative and integral}
\end{aligned}$$

Draw a table for the two-phase method.

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 1 | 0 | $-M$ | |
| $x_4$ | 0 | 2 | 3 | 3 | 1 | 0 | 11 |
| $x_5$ | $-M$ | 0 | 1 | 0 | 0 | 1 | 4 |
| | | -1 | -2 | -1 | 0 | 0 | 0 |
| | | 0 | -1 | 0 | 0 | 0 | -15 |

Change $x_4$ for $x_2$ in the basic variables.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | |
|---|---|---|---|---|---|---|
| $x_2$ | $\frac{2}{3}$ | 1 | 1 | $\frac{1}{3}$ | 0 | $1\frac{1}{3}$ |
| $x_5$ | $-\frac{2}{3}$ | 0 | -1 | $-\frac{1}{3}$ | 1 | $\frac{1}{3}$ |
| | $\frac{1}{3}$ | 0 | 1 | $\frac{2}{3}$ | 0 | $\frac{22}{3}$ |
| | $\frac{2}{3}$ | 0 | 1 | $\frac{1}{3}$ | 0 | $-\frac{34}{3}$ |

The procedure has ended, but there still remains a non-zero artificial variable $x_5$ among the basic variables, therefore no solutions exist for Programme 3.

(3) No solution

(1) $z^* = \frac{22}{3}$, $\left(0, \frac{11}{3}\right)$

(2) $z^* = 7$, $(1, 3)$

Therefore the solution is $x_1^* = 1$, $x_2^* = 3$, $x_3^* = x_4^* = x_5^* = 0$, and $z^* = 7$.

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**Algorithm 2** *Gomory algorithm for integer programming.*

**while** solution not wholly all integers **do**
  **choose** one non-integral optimal approximation
  **write** a relation from the row where that variable is basic
  **rewrite** the relation to make all fractional coefficients some integer plus a proper fraction

**move** all the fractions to LHS, and all the non-fractions to RHS
**write** a new constraint as LHS $\geq 0$
**find** the solution for the original problem together with the new constraint
**endwhile**

**Example 2.** (*Problem 7.1; Bronson, 1982*)

$$\begin{aligned} \text{maximise:} \quad & z = 2x_1 + x_2 \\ \text{subject to:} \quad & 2x_1 + 5x_2 \leq 17 \\ & 3x_1 + 2x_2 \leq 10 \\ \text{with:} \quad & x_1, x_2 \text{ non-negative and integral} \end{aligned}$$

Use cut algorithm.

**Solve**

**Solution.**Find the first approximation of Programme 1 normally using the simplex method.

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | |
|---|---|---|---|---|---|---|
| | | 2 | 1 | 0 | 0 | |
| $x_3$ | 0 | 2 | 5 | 1 | 0 | 17 |
| $x_4$ | 0 | 3 | 2 | 0 | 1 | 10 |
| | | -2 | -1 | 0 | 0 | 0 |

Since $\frac{10}{3} < \frac{17}{2}$, we know that 3 is the pivot element, and therefore we replace the basic variable $x_4$ with $x_1$.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | |
|---|---|---|---|---|---|
| $x_3$ | 0 | $\frac{11}{3}$ | 1 | $-\frac{2}{3}$ | $\frac{31}{3}$ |
| $x_1$ | 1 | $\frac{2}{3}$ | 0 | $\frac{1}{3}$ | $\frac{10}{3}$ |
| | 0 | $\frac{1}{3}$ | 0 | $\frac{2}{3}$ | $\frac{20}{3}$ |

We have $x_1^* = \frac{10}{3}$, $x_3^* = \frac{31}{3}$, $x_2^* = x_4^* = 0$ and $z^* = \frac{20}{3}$. Since both $x_1^*$ and $x_3^*$ are non-integers, arbitrarily choose the former to generate a new constraint. Then our Programme 2 becomes,

$$\begin{aligned} x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_4 = \frac{10}{3} = 3 + \frac{1}{3} \\ \frac{2}{3}x_2 + \frac{1}{3}x_4 - \frac{1}{3} = 3 - x_1 \\ \frac{2}{3}x_2 + \frac{1}{3}x_4 - \frac{1}{3} \geq 0 \\ \frac{2}{3}x_2 + \frac{1}{3}x_4 \geq \frac{1}{3} \\ 2x_2 + x_4 \geq 1 \end{aligned}$$

and our new programme becomes

$$\begin{aligned} \text{maximise:} \quad & z = 2x_1 + x_2 + 0x_3 + 0x_4 \\ \text{subject to:} \quad & \frac{11}{3}x_2 - \frac{2}{3}x_4 = \frac{31}{3} \\ & x_1 + \frac{2}{3}x_2 - \frac{1}{3}x_4 = \frac{10}{3} \\ \text{with:} \quad & \text{all variables non-negative and integral} \end{aligned}$$

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | |
|---|---|---|---|---|---|---|---|---|
| | | 2 | 1 | 0 | 0 | 0 | $-M$ | |
| $x_1$ | 0 | 1 | $\frac{2}{3}$ | 0 | $\frac{1}{3}$ | 0 | 0 | $\frac{10}{3}$ |
| $x_3$ | 0 | 0 | $\frac{11}{3}$ | 1 | $-\frac{2}{3}$ | 0 | 0 | $\frac{31}{3}$ |
| $x_6$ | $-M$ | 0 | 2 | 0 | d1 | -1 | 1 | 1 |
| | | -2 | -1 | 0 | 0 | 0 | 0 | 0 |
| | | 0 | -2 | 0 | -1 | 1 | -1 | -1 |

Now $x_2$ replaces $x_6$ in the basic variables and becomes the pivot element.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 0 | 0 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 3 |
| $x_3$ | 0 | 0 | 1 | $-\frac{15}{6}$ | $\frac{11}{6}$ | $-\frac{11}{6}$ | $\frac{17}{2}$ |
| $x_2$ | 0 | 1 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| | -2 | 0 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

This becomes,

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | |
|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 0 | 0 | $\frac{1}{3}$ | 3 |
| $x_3$ | 0 | 0 | 1 | $-\frac{5}{2}$ | $\frac{11}{6}$ | $\frac{17}{2}$ |
| $x_2$ | 0 | 1 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |
| | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{13}{2}$ |

Then our first approximation of Programme 2 is $x_1^* = 3$, $x_2^* = \frac{1}{2}$, $x_3^* = \frac{17}{2}$, $x_4^* = x_5^* = 0$, and $z^* = \frac{13}{2}$. Arbitrarily choose $x_2^*$ to generate the new constraint.

$$\begin{aligned}
x_2 + \frac{1}{2}x_4 - \frac{1}{2}x_5 &= \frac{1}{2} \\
\frac{1}{2}x_4 - \frac{1}{2}x_5 - \frac{1}{2} &= -x_2 \\
\frac{1}{2}x_4 - \frac{1}{2}x_5 - \frac{1}{2} &\geq 1 \\
x_4 - x_5 &\geq 1
\end{aligned}$$

Then our Programme 3 becomes,

$$\begin{aligned}
\text{maximise:}\quad & z = 2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 \\
\text{subject to:}\quad & x_1 + \frac{1}{3}x_5 = 3 \\
& x_3 - \frac{5}{2}x_4 + \frac{11}{6}x_5 = \frac{17}{2} \\
& x_2 + \frac{1}{2}x_4 - \frac{1}{2}x_5 = \frac{1}{2} \\
& x_4 - x_5 \geq 1 \\
\text{with:}\quad & \text{all variables non-negative and integral}
\end{aligned}$$

We draw our table for this programme.

| | | $x_1$<br>2 | $x_2$<br>1 | $x_3$<br>0 | $x_4$<br>0 | $x_5$<br>0 | $x_6$<br>0 | $x_7$<br>$-M$ | |
|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | 0 | 1 | 0 | 0 | 0 | $\frac{1}{3}$ | 0 | 0 | 3 |
| $x_2$ | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ |
| $x_3$ | 0 | 1 | 0 | 0 | $-\frac{5}{2}$ | $\frac{11}{6}$ | 0 | 0 | $\frac{17}{2}$ |
| $x-7$ | $-M$ | 0 | 0 | 0 | 1 | -1 | -1 | 1 | 1 |
| | | -2 | -1 | 0 | 0 | 0 | 0 | 0 | 0 |
| | | 0 | 0 | 0 | -1 | 1 | 1 | -1 | -1 |

Then $x_4$ replaces the basic $x_7$ to become the pivot element.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 0 | 0 | $\frac{1}{3}$ | 0 | 3 |
| $x_2$ | 0 | 1 | 0 | 0 | 0 | $\frac{1}{2}$ | 0 |
| $x_3$ | 1 | 0 | 0 | 0 | $-\frac{2}{3}$ | $-\frac{5}{2}$ | 11 |
| $x_4$ | 0 | 0 | 0 | 1 | -1 | -1 | 1 |
| | -2 | -1 | 0 | 0 | 0 | 0 | 0 |

Next, $x_1$ remains basic and becomes a pivot element.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 0 | 0 | $\frac{1}{3}$ | 0 | 3 |
| $x_2$ | 0 | 1 | 0 | 0 | 0 | $\frac{1}{2}$ | 0 |
| $x_3$ | 0 | 0 | 0 | 0 | -1 | $-\frac{5}{2}$ | 8 |
| $x_4$ | 0 | 0 | 0 | 1 | -1 | -1 | 1 |
| | 0 | -1 | 0 | 0 | $\frac{1}{3}$ | 0 | 6 |

This becomes

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 0 | 0 | $\frac{1}{3}$ | 0 | 3 |
| $x_2$ | 0 | 1 | 0 | 0 | 0 | $\frac{1}{2}$ | 0 |
| $x_3$ | 0 | 0 | 0 | 0 | -1 | $-\frac{5}{2}$ | 8 |
| $x_4$ | 0 | 0 | 0 | 1 | -1 | -1 | 1 |
| | 0 | 0 | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{2}$ | 6 |

The optimum point for Programme 3 is then, $x_1^* = 3$, $x_3^* = 8$, $x_4^* = 1$, $x_2^* = x_5^* = x_6^* = 0$ and $z^* = 6$. Therefore the solution to the original problem Programme 1 is $x_1^* = 3$, $x_2^* = 0$ at the objective value $z^* = 6$.

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**Definition 4.** A *transportation problem* involves $m$ *sources* each of which supplies $a_i$, $i = 1, \ldots, m$, units of a certain product, and $n$ *destinations* each of which requires $b_i$, $i = 1, \ldots, n$, units of the same. The problem may be stated as following.

$$\begin{aligned} \text{maximise:} \quad & z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ \text{subject to:} \quad & \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \ldots, m \\ & \sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n \\ \text{with:} \quad & \text{all } x_{ij} \text{ non-negative and integral} \end{aligned}$$

The total supply and the total demand are assumed to be equal. Were this not so, a fictitious destination or a fictitious source is added.

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**Definition 5.** The *north-west corner rule* finds an initial basic solution for the transportation algorithm of the integer programming. It begins with the (1,1) cell in the $m \times n$ table, and allocates as many units as possible to $x_{11}$ violating neither the constraints of supply, that is the summation along each row, nor those of demand, that is the summation along each column. Then carry on moving for each step either right or downwards, until we reach the lower-right corner, $x_{mn}$.

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**Definition 6.** A *loop*, which is a sequence of cells in the table used for finding the solution in the transportation problem, has the following properties.

a. each pair of consecutive cells is on either the same row or the same column

b. no three, or in fact any odd-numbered, consecutive cells lie in the same row or column
c. the first and the last cells are on the same row or column
d. the path along the loop is self-avoiding, that is no cells appear more than once in the sequence

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**Algorithm 3** *Transportation algorithm.*

**while** optimal solution not attained **do**
  **find** an initial, basic feasible solution using, for instance, the North-west corner rule
  **let** *either* $u_i = 0$ *or* $v_j = 0$ *depending on whether the* $i^{\text{th}}$*-row or the* $j^{\text{th}}$*-column*
    has the maximum number of basic solutions
  **find** all $u_i$ and $v_j$, $i = 1, \ldots, m$ and $j = 1, \ldots, n$ from $u_i + v_j = c_{ij}$ for basic variables,
    and from $c_{ij} - u_i - v_j$ for non-basic variables
  **improve** the solution
**endwhile**

**Note 1.** In a transportation problem, optimal solution is achieved when $c_{ij} - u_i - u_j \geq 0$ for all transportation costs per unit $c_{ij}$ of all non-basic variables.

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## Bibliography

Richard Bronson. *Theory and problems of operations research.* Schaum's outline series, McGraw-Hill, Singapore, 1982 (1983)